Gas clouds rotating around a principal axis: II. The separation of variables

2001 J. Phys. A: Math. Gen. 34 9195

(http://iopscience.iop.org/0305-4470/34/43/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 9195–9209 PII: S0305-4470(01)25094-X

Gas clouds rotating around a principal axis: II. The separation of variables

B Gaffet

CNRS and CEA DSM/DAPNIA/Service d'Astrophysique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 27 April 2001
Published 19 October 2001
Online at stacks.iop.org/JPhysA/34/9195

Abstract

We present a new formulation of the spinning gas cloud model of Ovsiannikov and Dyson (in the case of rotation around a fixed axis), in terms of a 4×4 symmetric matrix $\mathcal{M}$ which is a linear function of four coordinates x^i. We obtain a pair of separable variables ℓ_1, ℓ_2 related to the x^i by a simple transformation, and discuss their geometrical meaning.

PACS numbers: 02.30.Ik, 02.30.Hq, 51.10.+y

1. Introduction

We consider the ordinary differential reduction of the equations of gas dynamics (Ovsiannikov 1956, Dyson 1968) which represents rotating gas clouds of ellipsoidal shape, expanding adiabatically into the vacuum while maintaining a Gaussian density profile, a linear velocity distribution, and a uniform temperature inside the cloud.

The spatial distributions thus being specified, the evolution of the system may be described by functions of time only, which satisfy a differential system of order 18. That system may equivalently be viewed as representing Hamiltonian motion of a particle in a potential in nine-dimensional Euclidean space (Dyson 1968).

We restrict our consideration here to the case of a monatomic gas (of adiabatic index $\gamma = 5/3$) evolving without vorticity; the equivalent single-particle description then represents Hamiltonian motion in the space $O(3) \times S_2$, where S_2 is the unit 2-sphere (where each point corresponds to a particular ellipsoidal shape) and $O(3)$ is the three-dimensional rotation group (where each point corresponds to a particular orientation of the ellipsoidal cloud).

In more general cases including vorticity, there is a second symmetry group $O(3)$, and the Hamiltonian motion takes place in $O(3) \times S_2 \times O(3)$, which is also the eight-dimensional sphere S_8 (the unit sphere centred at the origin of coordinates in Dyson's nine-dimensional Euclidean space), but we will not consider such motions in the present paper.

In a recent work (Gaffet 2000, hereafter referred to as paper I), we have shown Liouville integrability of the Hamiltonian motion in the restricted case of a cloud rotating around a fixed principal axis (this result was later extended to cover fully general states of rotation by Gaffet (2001)). In such cases the motion takes place in $O(2) \times S_2$ and, as the three integrals of motion required for Liouville integrability do not involve the position coordinate in the space $O(2)$, the system effectively represents particle motion on the sphere S_2 in a potential.

Liouville integrable systems (Landau and Lifshitz 1960) are characterized by the property that the differential of action, $\mathrm{d}S = p\,\mathrm{d}q$, is an exact differential on all surfaces (S) in phase space (the Liouville tori) obtained by specifying the values of a complete set of commuting integrals of motion. In the present case, these are a family of surfaces (which we may refer to as the two-dimensional phase space) in three-dimensional space R^3, whose equation has been given in paper I for all values of the three integrals of motion (the energy $\hat{E}$, the total angular momentum $\vec{J}^2$, and another integral denoted by I_6). However, the quadratures determining the action cannot be performed in practice unless a parametrization of the corresponding surface is available.

It is the purpose of the present paper to provide such a parametrization, and, as it turns out, the parametrization that we obtain also solves the problem of separating the variables at the same time.

In the simpler non-rotating case (Gaffet 1998) we have shown, through an appropriate change of variables, that the two-dimensional phase space can be represented by a surface which is of the fourth degree only (a quartic surface). We show in section 2 that a corresponding change of variables (leading to a quartic representation of S) also exists in cases of non-vanishing angular momentum. This constitutes an important simplification, in view of the fact that the difficulty of parametrizing a surface increases rapidly with its degree.

As a second step (section 3) we remark that the quartic surfaces obtained present a certain number of singularities (*conic points*) which are associated with the *stationary points* of the spherical motion on S_2 (points where the velocity vanishes, but in general the acceleration does not). The parametrization problem is solved by cutting the surface by arbitrary straight lines passing through one such conic point, i.e. by selecting as independent variables the parameters defining the direction of these straight lines. (This amounts to a linear transformation in a homogeneous coordinate system.)

From each line through a conic point, *two* tangent planes at the conic point may be drawn since the tangent cone must be of the second degree. The tangent planes form a one-parameter family, with parameter ℓ, say. The two tangent planes issuing from a line thus determine a pair of values $(\ell_1;\ \ell_2)$, which turn out to be *separable variables*, meaning that the differential system governing their evolution has a manifestly separable form.

2. Generalizing from non-rotating to rotating cases

2.1. The general form of the system

In an earlier paper (Gaffet 2001) we have shown that the equations of motion of the expanding and rotating cloud in the case of vanishing vorticity may be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}u} \ln \Delta_i = 2\, v_{ii} \qquad (i = 1, 2, 3) \tag{2.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}u} + v^2 - \frac{1}{3}\,\mathrm{Tr}\,(v^2) = [\omega, v] + \Delta^{-1} - \frac{1}{3}\,\mathrm{Tr}\,(\Delta^{-1}) \tag{2.2}$$

where Δ is a diagonal matrix of unit determinant, v a symmetric 3×3 matrix and u the independent variable. The antisymmetric matrix ω is fixed by the off-diagonal part of v as

$$\omega_{ij} = \left(\frac{\Delta_i + \Delta_j}{\Delta_i - \Delta_j}\right) v_{ij} \qquad (i \neq j). \tag{2.3}$$

A related formulation, in terms of the permutation invariant quantities

$$\begin{aligned} X_n &= \mathrm{Tr}\,(\Delta\, v^n) \qquad (n = 0, 1, 2) \\ Y_n &= \mathrm{Tr}\,(\Delta^{-1}\, v^n) \\ T &= -\tfrac{1}{2}\,\mathrm{Tr}\,(v^2) \\ P &= \det\,(v) \end{aligned} \tag{2.4}$$

was found to be especially useful for expressing compactly the integrals of motion; it is

$$\begin{aligned} T'(u) &= 3P - Y_1 \\ P'(u) &= -\tfrac{2}{3}T^2 + (\tfrac{2}{3}\,T\,Y_0 + Y_2) \\ X_0'(u) &= 2\,X_1 \\ X_1'(u) &= \left(X_2 - \frac{2}{3}\,T\,X_0\right) + \left(3 - \frac{X_0 Y_0}{3}\right) \\ X_2'(u) &= -\tfrac{4}{3}\,T\,X_1 - \tfrac{2}{3}\,Y_0 X_1 \\ Y_0'(u) &= -2\,Y_1 \\ Y_1'(u) &= -\left(3\,Y_2 + \frac{2}{3}\,T\,Y_0\right) + 2\left(\frac{Y_0^2}{3} - X_0\right) \\ Y_2'(u) &= 4\,(\tfrac{2}{3}\,T\,Y_1 - P\,Y_0) + 2(\tfrac{2}{3}\,Y_0 Y_1 + X_1). \end{aligned} \tag{2.5}$$

The energy constant $\hat{E}$ ($\hat{E} = 9m/2$) and the total angular momentum $\vec{J}^2$ (also denoted by α^2) then assume the following simple forms:

$$\begin{aligned} 9m &\equiv 2\hat{E} = (X_0 X_2 - X_1^2) + 3\,X_0 \\ \alpha^2 &\equiv \vec{J}^2 = (X_0 X_2 - X_1^2) + (3\,Y_2 + 4\,T\,Y_0). \end{aligned} \tag{2.6}$$

A third integral I_6 and a fourth integral L_6 were also determined in this paper.

2.2. *The block-diagonal reduction*

The system (2.1), (2.2) admits a reduction where the matrix v remains block-diagonal at all times; this corresponds physically to a cloud rotating around a fixed principal axis (which we take, without loss of generality, to be the third axis). This case was investigated in paper I, using a different system of notation; it is thus of interest to establish the correspondence between the two.

In this earlier work the role of the diagonal matrix Δ (which may be viewed as parametrizing the unit 2-sphere, where the equivalent single-particle Hamiltonian motion takes place) was played by a pair of variables π and X:

$$\begin{aligned} \pi &\equiv \frac{1}{\Delta_3} \\ X &\equiv \frac{(\Delta_1 + \Delta_2)}{\Delta_3} \end{aligned} \tag{2.7}$$

and we note that

$$\begin{aligned} X_0 &\equiv \mathrm{Tr}\,(\Delta) \equiv (X+1)/\pi \\ Y_0 &\equiv \mathrm{Tr}\,(\Delta^{-1}) \equiv (X + \pi^3)/\pi^2. \end{aligned} \tag{2.8}$$

A 3-vector $\vec{\xi} \equiv (\xi,\, \eta,\, \zeta)$ was introduced, representing the derivative of Δ (i.e. the diagonal part v_0 of v: see equation (2.1)), in the following way:

$$\begin{aligned} \sqrt{3}\,\xi &= (v_{22} - v_{33})/\Delta_1 \\ \sqrt{3}\,\eta &= (v_{33} - v_{11})/\Delta_2 \\ \sqrt{3}\,\zeta &= (v_{11} - v_{22})/\Delta_3 \end{aligned} \tag{2.9}$$

and, conversely,

$$\begin{aligned}\sqrt{3}\, v_{11} &= (\zeta\, \Delta_3 - \eta\, \Delta_2)\\ \sqrt{3}\, v_{22} &= (\xi\, \Delta_1 - \zeta\, \Delta_3)\\ \sqrt{3}\, v_{33} &= (\eta\, \Delta_2 - \xi\, \Delta_1).\end{aligned} \tag{2.10}$$

It was also shown that a certain variable ρ played an important role:

$$-(\rho + \pi) = (v^2)_{33} + \frac{T}{3}. \tag{2.11}$$

(In particular, the line $\rho = 0$ was shown by Gaffet (1998) to be a line of singularity of a certain surface (S) in phase space; see below, section 2.4).

In paper I the expression of the integral I_6 (also denoted 4ε) was found for the block-diagonal case:

$$4\,\varepsilon \equiv I_6 = \Omega^2 + 4\, f(\rho\, \Delta_1 + 1)(\rho\, \Delta_2 + 1) \tag{2.12}$$

where

$$\begin{aligned}\Omega &= \left(\xi + \eta - \frac{\rho}{\pi}\, \zeta\right)\\ f &= \frac{\alpha^2\pi/3}{(X^2 - 4\,\pi^3)}.\end{aligned} \tag{2.13}$$

In terms of the new variables, the quantity f is

$$f \equiv \frac{(T_0 - T)}{3\,\pi} \tag{2.14}$$

where

$$\begin{aligned}-\,2\,T_0 &\equiv \mathrm{Tr}\,(v_0^2) = (v_{11}^2 + v_{22}^2 + v_{33}^2)\\ -\,2\,T\ &\equiv \mathrm{Tr}\,(v^2)\end{aligned} \tag{2.15}$$

and the expression of T_0 in terms of the other variables has been given by Gaffet (2001).

2.3. *The variable W*

Fixing the values of the three integrals of motion m, ε and α^2 results in an algebraic relation $F(X; \pi; \rho) = 0$ (equation (6.6) of paper I), representing a two-dimensional surface (S) in phase space, where all three integrals remain constant. In the non-rotating case where the matrix v is diagonal, we found (Gaffet 1998) that the variables $(\pi; \rho; \zeta)$ were particularly suitable for representing that surface. However, when rotation is included, the variable ζ requires some appropriate generalization.

In paper I, we found that the equation of S involved the variable ζ only through its square ζ^2 and the product $\zeta\,\Omega$. Remembering definition (2.9) of ζ, we remark that its square:

$$\zeta^2 = -\pi^2(v_{33}^2 + \tfrac{4}{3}\, T_0)$$

involves a characteristic coefficient of the diagonal part v_0 of v, rather than of v itself. Thus the natural generalization of ζ^2 in rotating cases should be

$$Z \equiv (\zeta^2 + 4\,\pi^3 f) \equiv \pi^2(\rho + \pi - T) \equiv -\,\pi^2[v_{33}^2 + \tfrac{4}{3}\, T]. \tag{2.16}$$

The product $\zeta\,\Omega$:

$$\zeta\,\Omega = \left(\xi\,\zeta + \eta\,\zeta - \frac{\rho}{\pi}\,\zeta^2\right) \tag{2.17}$$

involves, in addition to ζ^2, the quantities

$$
\begin{aligned}
-\frac{\eta\,\zeta}{\Delta_1} &\equiv (v^2)_{11} + T - \frac{2}{3}\,T_0 \\
-\frac{\xi\,\zeta}{\Delta_2} &\equiv (v^2)_{22} + T - \frac{2}{3}\,T_0
\end{aligned}
\tag{2.18}
$$

which admit the following block-diagonal natural generalization:

$$
\begin{aligned}
-\frac{1}{\Delta_1}\left(\eta\,\zeta - 2\,f\,\frac{\Delta_1}{\Delta_3}\right) &\equiv (v^2)_{11} + \frac{T}{3} \\
-\frac{1}{\Delta_2}\left(\xi\,\zeta - 2\,f\,\frac{\Delta_2}{\Delta_3}\right) &\equiv (v^2)_{22} + \frac{T}{3}.
\end{aligned}
\tag{2.19}
$$

Therefore the variable generalizing the product $\zeta\,\Omega$ is

$$
\begin{aligned}
-W &= \left(\eta\,\zeta - 2\,f\,\frac{\Delta_1}{\Delta_3}\right) + \left(\xi\,\zeta - 2\,f\,\frac{\Delta_2}{\Delta_3}\right) - \frac{\rho}{\pi}\,(\zeta^2 + 4\,\pi^3 f) \\
&= \zeta\,\Omega - 2\,f\,(X + 2\,\pi^2\rho).
\end{aligned}
\tag{2.20}
$$

This leads to a new formulation, replacing equations (6.4) and (6.5) of paper I, of the surface (S):

$$
\left(\frac{\alpha^2}{3} - 3\,m\right) = \frac{1}{\pi^2}\,[\rho\,X + (\rho - \pi)\,Z] - \frac{(W+1)}{\pi}
\tag{2.21a}
$$

$$
\left(4\,\varepsilon + \frac{\alpha^2}{3\,\pi^2}\right) = \frac{W}{\pi^3}\,(X + 2\,\pi^2\rho) - \frac{(\rho+\pi)}{\pi^4}(X^2 - 4\,\pi^3) - \frac{Z}{\pi^3}\left(\frac{\rho\,X}{\pi} + \pi\,\rho^2 + 1\right)
\tag{2.21b}
$$

$$
\varepsilon\,Z + \frac{\alpha^2}{3}(\rho + \pi) = \frac{W^2}{4}.
\tag{2.21c}
$$

In terms of π, ρ and of the permutation invariants (2.4), W is also

$$
\begin{aligned}
W &= \left(X_2 + \frac{T}{3}\,X_0\right) + \frac{[\rho\,Z + (\rho + \pi)]}{\pi} \\
&= X_2 + T\left(\frac{X_0}{3} - \pi\,\rho\right) + \frac{(\rho + \pi)}{\pi}\,(\pi^2\rho + 1).
\end{aligned}
\tag{2.22}
$$

2.4. The singular line $\rho = 0$

When $\rho = 0$, system (2.21) reduces to

$$
\left(3\,m - \frac{\alpha^2}{3}\right)\pi = W + 1 + Z
\tag{2.23a}
$$

$$
\left(4\,\varepsilon + \frac{\alpha^2}{3\,\pi^2}\right) = \frac{X}{\pi^3}\,(W - X) + 4 - \frac{Z}{\pi^3}
\tag{2.23b}
$$

$$
\left(\varepsilon\,Z + \frac{\alpha^2\pi}{3}\right) = \frac{W^2}{4}.
\tag{2.23c}
$$

The first and last equations determine π and Z linearly in terms of W:

$$
4\,\beta\,\pi = \Phi\,(W) \equiv (W^2 + 4\,\varepsilon\,W + 4\,\varepsilon)
\tag{2.24}
$$

where

$$
\beta = 3\,\varepsilon\,m + \frac{\alpha^2}{3}\,(1 - \varepsilon)
\tag{2.25}
$$

and the remaining equation then determines a pair of distinct solutions for X. This means that the locus specified by the condition $\rho = 0$ is a line of intersection of two sheets of the surface (S). As usual the singularity can be removed through the introduction of a variable (denoted R) which reduces to the slopes of the tangent planes near the singularity:

$$R = \frac{\lfloor 4\beta\pi - \Phi(W)\rfloor}{\rho}. \tag{2.26}$$

It is worth noting that R is related to the variable X, as

$$-\pi R = 4\varepsilon X + W^2 - \tfrac{4}{3}\alpha^2\rho \tag{2.27}$$

which we may also write in the form

$$-R = 4\varepsilon(\Delta_1 + \Delta_2) + (W^2 - \tfrac{4}{3}\alpha^2\rho)\,\Delta_3. \tag{2.28}$$

The main advantage of introducing the variable R is that, while the surface (S) is of degree 6 in coordinates (ρ; π; W), it becomes a quartic surface in coordinates (ρ; R; W).

It is of interest to consider the algebraic nature of the curve $\rho = 0$ in more detail. As already mentioned, equation (2.23b) is of second degree in X:

$$X^2 - XW + [4\pi^3(\varepsilon - 1) + 3m\pi - (W + 1)] = 0 \tag{2.29}$$

and its algebraic genus is governed by its discriminant, which is a sixth-degree polynomial in W:

$$(\varepsilon - 1)\,A_6(W) = 4\pi^3(\varepsilon - 1) + 3m\pi - \frac{(W + 2)^2}{4} \tag{2.30}$$

(where π is given versus W by equation (2.24)). An alternative equivalent form is

$$\varepsilon A_6(W) = 4\varepsilon\pi^3 + \frac{\alpha^2\pi}{3} - \frac{W^2}{4} \tag{2.31}$$

and constitutes a straightforward generalization of the corresponding result (equation (3.2) or (4.4)) obtained by Gaffet (1998). Whenever $A_6(W)$ has a double root the curve $\rho = 0$ is of genus 1, and can be parametrized by elliptic functions.

A reduced form of A_6 may be obtained through the linear transformation (assuming $0 < \varepsilon < 1$)

$$(W + 2\varepsilon) = 2w\sqrt{\varepsilon(1 - \varepsilon)} \tag{2.32}$$

which eliminates all odd powers of the independent variable, except the linear one:

$$\hat{\beta}^3 A_6 = 4(w^2 + 1)^3 + \hat{\beta}^2\left(\frac{\alpha^2}{3} - 3m\right)(w^2 + 1) + P_1(w) \tag{2.33}$$

where

$$P_1(w) = 2\hat{\beta}^3\Big[w\sqrt{\varepsilon(1 - \varepsilon)} + (\tfrac{1}{2} - \varepsilon)\Big]$$

and

$$\hat{\beta} = \frac{\beta}{\varepsilon(1 - \varepsilon)}.$$

In cases where ε is negative or greater than 1 on the other hand, letting

$$(W + 2\varepsilon) = 2w\sqrt{\varepsilon(\varepsilon - 1)} \tag{2.34}$$

the reduced form is given by

$$\hat{\beta}^3 A_6 = 4(w^2 - 1)^3 + \hat{\beta}^2\left(\frac{\alpha^2}{3} - 3m\right)(w^2 - 1) + P_1(w) \tag{2.35}$$

where

$$P_1(w) = 2\,\hat{\beta}^3\Big[w\,\sqrt{\varepsilon\,(\varepsilon - 1)} + (\tfrac{1}{2} - \varepsilon)\Big]$$

and

$$\hat{\beta} = \frac{\beta}{\varepsilon\,(\varepsilon - 1)}.$$

2.5. A new formulation

For the derivatives of the new variables $(\pi;\rho;W;R)$ we obtain expressions which are straightforward generalizations of those (equations (2.31)–(2.34)) found by Gaffet (1998) for the case without rotation:

$$\begin{aligned}
&[2\,\pi\,\rho'(u) - \rho\,\pi'(u)] = \frac{R'(u)}{4\,\varepsilon}\\
&[2\,\pi\,R' - R\,\pi'] = 2\,W W' - \frac{4\,\alpha^2}{3}\,\rho'\\
&[2\,(\rho + \pi)\,W' - W\,\rho'] = \Phi'\,(W)\,\frac{\pi'}{2}\\
&\Phi'\,(W)\,W' + 4\,\varepsilon\,(R\,\rho' + \rho\,R') = 4\,\beta\,\pi'
\end{aligned} \tag{2.36}$$

(where the last equation results from differentiation of equation (2.26)). These four simple relations between derivatives can be arranged compactly in matrix form. Introducing

$$x_1 \equiv W \qquad x_2 \equiv \rho \qquad x_3 \equiv R \qquad x_4 \equiv \pi$$

the system is

$$\mathcal{M}\,[\overrightarrow{x'}\,] = 0 \tag{2.37}$$

where $[\overrightarrow{x'}\,]$ is the column vector with components $x_i'(u)$, and $\mathcal{M}$ is *a* 4×4 *symmetric matrix, whose coefficients are linear functions of the four coordinates*

$$\mathcal{M} = \begin{bmatrix} -4\,(\rho + \pi) & 2\,W & 0 & \Phi'(W) \\ 2\,W & -4\,\alpha^2/3 & -2\,\pi & R \\ 0 & -2\,\pi & 1/(4\,\varepsilon) & \rho \\ \Phi'\,(W) & R & \rho & -4\,\beta \end{bmatrix}. \tag{2.38}$$

Clearly its determinant must be zero: together with equation (2.26), the condition

$$\det \mathcal{M} = 0$$

constitutes the equation of the surface (S). It is an algebraic surface of fourth degree in coordinates $(\rho;W;R)$.

The property of $\mathcal{M}$ to be symmetric is easily understood as a consequence of the fact that it is only determined up to multiplication by an arbitrary (regular) matrix on the left; multiplication by the appropriate rotation matrix will make $\mathcal{M}$ symmetric, in case it was not.

The cofactors C_{ij} of $\mathcal{M}$ must be proportional to the products $x_i' x_j'$. The proportionality factor may be easily determined from, e.g., $x_4'^2/C_{44}$; we have

$$x_4'(u) \equiv \pi'(u) = -2\,\pi\,v_{33}$$

where

$$v_{33}^2 = -\left[\rho + \pi + \frac{T}{3}\right]$$

whereas

$$\varepsilon\, C_{44} \;=\; 4\,(\rho\,+\,\pi)\,\left(4\,\varepsilon\,\pi^2\,+\,\frac{\alpha^2}{3}\right)\;-\;W^2.$$

Making use of equations (2.21*c*) and (2.16),

$$\frac{C_{44}}{4\,\pi^2}\;=\;\left[4\,(\rho\,+\,\pi)\;-\;\frac{Z}{\pi^2}\right]\;=\;3\left[\rho\,+\,\pi\,+\,\frac{T}{3}\right]\;=\;-\,3\,\upsilon_{33}^2.$$

The proportionality factor is thus an absolute constant:

$$\frac{C_{ij}}{x_i' x_j'}\;=\;\frac{C_{44}}{x_4'^2}\;=\;-\,3. \tag{2.39}$$

The new formulation in terms of the matrix $\mathcal{M}$ thus contains both *the equation of the surface (S), and the equations of evolution of the physical variables*. It also contains their solution, in the form of a quadrature, as will be shown in section 4.

For self-consistency the derivative $\frac{\mathrm{d}}{\mathrm{d}\,u}(\det\,\mathcal{M})$ should vanish on the surface (S) where $\det(\mathcal{M}) = 0$. Since $\mathcal{M}$ happens to be a linear function of the coordinates

$$\mathcal{M} = M_{,i}\,x^i + M_0 \qquad (i = 1, 2, 3, 4) \tag{2.40}$$

(where M_0 and $M_{,i}$ are constant matrices), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,u}\,(\det\,\mathcal{M})\;&=\;C_{ij}\,\frac{\mathrm{d}\,M_{ij}}{\mathrm{d}\,u}\;=\;C_{ij}M_{,k\,ij}x_k'(u)\\ &=\;-\,3\,x_i'x_j'x_k'\,M_{,k\,ij}\end{aligned}$$

(with summation implied over repeated indices). It is thus sufficient that the four matrices $M_{,i}$ satisfy the condition

$$M_{,kij}\;+\;M_{,ijk}\;+\;M_{,jki}\;=\;0$$

or, in compact form,

$$M_{(,kij)} = 0 \tag{2.41}$$

in order that $\frac{\mathrm{d}}{\mathrm{d}\,u}$ $(\det\,\mathcal{M})$ vanishes on the surface. That condition is, in fact, satisfied in the present problem.

3. Separation of variables

As seen in the preceding section, the equations of motion are fixed by the 4×4 matrix $\mathcal{M}$, which is a linear function of four coordinates, and involves three free parameters (the integrals of motion) m, ε and α^2. We now choose for definiteness the particular values

$$m = 6 \qquad \varepsilon = 4 \qquad \alpha^2 = 12$$

which are not expected to be special in any way, inasmuch as the discriminant A_6 (see equation (2.30)) then does not present any double roots. We expect that the method of resolution described below should remain valid for arbitrary values of all the parameters.

For the above values of ε and α^2, the value $m_0 = 5$ is special, since the polynomial A_6 then has a double root, and it probably constitutes the physical minimum of the energy (see paper I, section 7.4); therefore we selected a somewhat higher value.

3.1. The conic points

The surface (S) of

$$F(\rho;\, W;\, R) \equiv \det(\mathcal{M}) = 0 \tag{3.1}$$

is a quartic surface in the space $(\rho;\, W;\, R)$; the main difference with respect to non-rotating cases is that its dependence on ρ is now *cubic*, instead of merely quadratic—which makes it more difficult to parametrize.

The property of F to be quadratic in ρ resulted from the presence of a singular point at infinity in the direction of the ρ-axis—a conic point; it is thus natural to look for such conic singular points on (S) in rotating cases. They are the stationary points of the motion on the 2-sphere S_2 (parametrized by the diagonal matrix Δ), where $X_1 = 0 = Y_1$, i.e. the *diagonal* part of the matrix v vanishes, and accordingly the vector $\vec{\xi}$ and the velocities $\rho'(u)$, $W'(u)$, $R'(u)$ (but, in general, not the accelerations) also vanish; they may be found by requiring that the cofactors C_{ij} of $\mathcal{M}$ vanish. These conditions yield the following system (using the notation of paper I):

$$\begin{aligned} &(X^2 - 4\pi^3) = \frac{\alpha^2\pi/3}{\varepsilon}\,[X\theta + \pi\,\rho^2 + 1] = \frac{\alpha^2\pi/3}{(\theta + 1)} \\ &(X + 1)\,[X\theta + X + 1] = 3\,m\,\pi \end{aligned} \tag{3.2}$$

for three unknowns $X, \pi, \rho \equiv \pi\,\theta$. The first two conditions express respectively that $\Omega = 0$ (as a consequence of the vanishing of $\vec{\xi}$)) and that $f = (\theta + 1)$; and the third condition expresses the vanishing of the constant term in the second-degree equation (6.5) of paper I, as a result of the vanishing of ζ[1].

We choose at random the following conic point K_0 among the many solutions of this system:

$$(K_0)\begin{cases} \rho_0 = 0.280\,2187 \\ \pi_0 = 0.239\,4700 \\ X_0 = 0.704\,4976 \end{cases} \tag{3.3}$$

from which the corresponding values of W and R may be deduced:

$$\begin{aligned} W_0 &= 3.197\,240 \\ R_0 &= -71.0352. \end{aligned} \tag{3.4}$$

Straight lines through the point K_0 intersect the surface (S) at two points distinct from K_0: in other words, the linear transformation (in homogeneous coordinates)

$$\begin{aligned} \tilde{\rho} &= \frac{1}{(\rho - \rho_0)} \\ \tilde{W} &= \frac{(W - W_0)}{(\rho - \rho_0)} \\ \tilde{R} &= \frac{(R - R_0)}{(\rho - \rho_0)} \end{aligned} \tag{3.5}$$

produces a modified form of the quartic equation of (S):

$$\frac{F}{(\rho - \rho_0)^4} \equiv \tilde{F}\,(\tilde{W};\, \tilde{R};\, \tilde{\rho}) = 0 \tag{3.6}$$

[1] It must be noted that the conditions determining a conic point:

$$X_l = 0 = Y_l \qquad \text{or} \qquad C_{ij} = 0$$

do not entail the vanishing of the velocity vector ξ if there is no rotation: in such cases the configuration of the cloud becomes *axisymmetric* at a conic point ($\Delta_1 = \Delta_2$; $\xi = \eta \neq 0$) and only ζ vanishes.

where $\tilde{F}$ is still of the fourth degree, and is quadratic in $\tilde{\rho}$:

$$\tilde{F} \equiv A(\tilde{W}, \tilde{R})\,\tilde{\rho}^2 + B(\tilde{W}, \tilde{R})\,\tilde{\rho} + C(\tilde{W}, \tilde{R}) \tag{3.7}$$

generalizing a similar equation (equation (2.20)) of Gaffet (1998). (The precise form of the polynomials A, B and C is given in the appendix.)

The discriminant, $(B^2 - 4AC)$, is a sixth-degree polynomial in $\tilde{W}$ and $\tilde{R}$, and it shares with the discriminant of equation (2.20) the property of being *fully decomposable into a product of six linear factors*:

$$\tilde{R} - (p_i \tilde{W} + q_i) \qquad (i = 1, \ldots, 6). \tag{3.8}$$

3.2. Geometric interpretation of the new variables

Our earlier results (Gaffet 1998) suggest that the coefficients p_i and q_i should be related to the roots ℓ_i of the polynomial A_6 in the following way:

$$\begin{aligned} p_i &= \frac{P_2(\ell_i)}{R_2(\ell_i)} \\ q_i &= \frac{Q_2(\ell_i)}{R_2(\ell_i)} \end{aligned} \tag{3.9}$$

where P_2, Q_2, R_2 are quadratic polynomials. The latter condition ensures that the linear factors (3.8) of $B^2 - 4AC$ admit (for a reason that will become apparent later) an alternative expression as second-degree factors in ℓ_i.

The determination of P_2, Q_2 and R_2 involves 12 linear equations for eight free parameters; however, the correspondence between the roots ℓ_i and the linear factors (3.8) is not *a priori* known, which makes the problem non-linear and its solution more difficult. Introducing the variable

$$\sigma_i = \frac{(p_i - p_a)}{(q_i - q_a)}$$

(where a is a particular value of the index i), we remark that it is a homographic (Möbius) function of ℓ_i; in addition

$$\tau_i = \frac{(\sigma_i - \sigma_a)}{(\sigma_i - \sigma_b)}$$

where b is another particular value of the index i, must be proportional to $\frac{(\ell_i - \ell_a)}{(\ell_i - \ell_b)}$. This makes it easier to establish the correspondence between indices of ℓ_i and (p_i, q_i), and finally to determine the polynomials P_2, Q_2 and R_2. As already observed, the linear factors (3.8) are thus proportional to factors quadratic in ℓ_i:

$$(\ell_i^2 - \ell_i S + P)$$

where S and P are ratios of linear inhomogeneous functions of $\tilde{W}$ and $\tilde{R}$:

$$\begin{aligned} S &= N_1(\tilde{W}, \tilde{R})/N_3(\tilde{W}, \tilde{R}) \\ P &= N_2(\tilde{W}, \tilde{R})/N_3(\tilde{W}, \tilde{R}). \end{aligned} \tag{3.10}$$

Conversely

$$\begin{aligned} \tilde{W} &= U_1(S, P)/U_3(S, P) \\ \tilde{R} &= U_2(S, P)/U_3(S, P) \end{aligned} \tag{3.11}$$

where U_1, U_2, U_3 are linear inhomogeneous functions of S and P.

This suggests introducing a pair of variables ℓ_1, ℓ_2, the roots of the second-degree equation

$$(\ell^2 - \ell S + P) = 0 \tag{3.12}$$

i.e.

$$\begin{aligned}(\ell_1 + \ell_2) &= S(\tilde{W};\ \tilde{R}) \\ \ell_1 \ell_2 &= P(\tilde{W};\ \tilde{R}).\end{aligned} \tag{3.13}$$

In terms of the new variables, the coefficient A of the second-degree equation $\tilde{F} = 0$ is

$$A = k_1 \frac{(\ell_1 - \ell_2)^2}{U_3^2} \tag{3.14}$$

where k_1 is a numerical constant. As for the discriminant, it assumes the essentially separable form

$$(B^2 - 4AC) = k_4 \frac{m_1^2 m_2^2}{U_3^6} \tag{3.15}$$

where k_4 is another numerical constant, $m_i^2 = m^2(\ell_i)$ $(i = 1, 2)$, and m^2 coincides with the sixth-degree polynomial A_6 found in the analysis of the singular curve $\rho = 0$; in the present case

$$m^2(\ell) \equiv (4\ell^6 - 36\ell^4 - 3042\ell^2 - 13\,500\ell - 14\,283). \tag{3.16}$$

For each value of ℓ, equation (3.12) represents a straight line (Δ) in the $(\tilde{W},\ \tilde{R})$ plane, or a corresponding plane (Π) passing through the point K_0 at infinity; let us similarly denote (Δ_i) as the line, and (Π_i) as the plane corresponding to the root ℓ_i of m^2. The lines (Δ) in the $(\tilde{W},\ \tilde{R})$ plane envelop a second-degree curve

$$(S^2 - 4P) = 0$$

which is the locus where $\ell_1 = \ell_2$, and, in view of equation (3.14), is also the trace in that plane of the tangent cone to the surface (S) at the conic point K_0. In other words, the family of tangent planes at K_0 may be rationally parametrized by ℓ, defined by equation (3.12). This leads us to the following simple geometrical interpretation of the new variables.

Through each point in the $(\tilde{W},\ \tilde{R})$ plane, or each straight line passing through K_0, one can draw a pair of tangent planes to (S) at K_0; the parameters of these two planes are respectively ℓ_1 and ℓ_2.

We show in the next section that choosing $(\ell_1,\ \ell_2)$ as dependent variables makes the differential system under study manifestly separable.

When $\ell = \ell_i$ is a root of m^2, any line passing through K_0 in the plane (Π_i) is tangent to (S), as a result of the vanishing of the discriminant $B^2 - 4AC$ (see equation (3.15)). Thus the plane (Π_i) remains tangent to (S) all along a curve (κ_i), which is a second-degree curve (a conic section) since (S) is of degree 4. Two such planes (Π_i) and (Π_j) intersect on a line, (Δ_{ij}) say, and the two corresponding conic sections (κ_i) and (κ_j), which both pass through K_0, must intersect a second time at some point K_{ij} on the line (Δ_{ij}). (The reason for that is that (Δ_{ij}) is tangent to (S) at each intersection with (κ_i) or (κ_j), so that each such intersection is double; but (Δ_{ij}) cannot have more than four intersections in all with (S).) Clearly, K_{ij} must be a conic point of the surface (S), since there exist two distinct tangent planes (Π_i) and (Π_j) at that point.

Thus the algebraic system (3.2) for the conic points presents the interesting particularity that it can be easily solved, once one arbitrary particular solution (such as K_0) has been obtained.

3.3. The separation of variables

The new variables S and P may be directly expressed in terms of the original variables (W, R, ρ), in a form analogous to equation (3.10):

$$\begin{aligned} S &= H_1(W, R, \rho)/H_3(W, R, \rho) \\ P &= H_2(W, R, \rho)/H_3(W, R, \rho) \end{aligned} \tag{3.17}$$

where H_1, H_2, H_3 are linear inhomogeneous functions. Through differentiation the derivatives $S'(u)$, $P'(u)$ may be obtained, using the values $W'(u)$, $R'(u)$, $\rho'(u)$ deduced from the cofactors of $\mathcal{M}$. It is found that they satisfy equations (4.10) of Gaffet (1998):

$$\begin{aligned} (S\,S'^2 - 2\,P'S') &= \frac{1}{k}\,F\,(S, P) \\ (P\,S'^2 - P'^2) &= \frac{1}{k}\,G\,(S, P) \end{aligned} \tag{3.18}$$

where k is a numerical constant; F and G are the polynomials defined generally by equations (4.11) and (4.12) therein, in terms of *a priori* arbitrary coefficients $a_0, \ldots, a_6$ which, in the present case, must be chosen to coincide with the coefficients of m^2 (see equation (3.16)). This shows that the equations of evolution of ℓ_1 and ℓ_2 have the separable form (equation (4.1) in Gaffet (1998))

$$\begin{aligned} \sqrt{k}\,\ell_1'(u) &= \frac{m_1}{(\ell_1 - \ell_2)} \\ \sqrt{k}\,\ell_2'(u) &= \frac{-m_2}{(\ell_1 - \ell_2)}. \end{aligned} \tag{3.19}$$

(Let us remark that, since k is negative in the present case ($k = -45$), real solutions are restricted to the range of values of ℓ for which the polynomial m^2 takes negative values.)

4. The integrating factor

The separable form (3.19) of the system is immediately integrable by quadrature in the form

$$\sqrt{k}\,\mathrm{d}\Phi = \frac{\mathrm{d}\ell_1}{m_1} + \frac{\mathrm{d}\ell_2}{m_2} \tag{4.1}$$

where Φ is the integration constant. We may rewrite it as

$$\mathrm{d}\Phi = \frac{(\ell_1 - \ell_2)}{m_1 m_2}\,(\ell_1'\,\mathrm{d}\ell_2 - \ell_2'\,\mathrm{d}\ell_1) \tag{4.2}$$

or, in coordinates S, P,

$$\mathrm{d}\Phi = \frac{\left(S'\,\mathrm{d}P - P'\,\mathrm{d}S\right)}{m_1 m_2}. \tag{4.3}$$

Transforming back to coordinates $(\tilde{W}, \tilde{R})$, we have

$$(\tilde{W}'\mathrm{d}\tilde{R} - \tilde{R}'\mathrm{d}\tilde{W}) = \frac{\partial\,(\tilde{W}, \tilde{R})}{\partial\,(S, P)}\,(S'\,\mathrm{d}P - P'\,\mathrm{d}S) \tag{4.4}$$

where, as a consequence of the form of the transformation formulae (3.11),

$$U_3^3\,\frac{\partial\,(\tilde{W}, \tilde{R})}{\partial\,(S, P)} = U_1\,\frac{\partial\,(U_2, U_3)}{\partial\,(S, P)} + \text{(circular permutation)} \tag{4.5}$$

is a constant (k_5, say). Hence

$$k_5\,\mathrm{d}\Phi = \frac{U_3^3}{m_1 m_2}\,(\tilde{W}'\mathrm{d}\tilde{R} - \tilde{R}'\mathrm{d}\tilde{W}). \tag{4.6}$$

Taking account of equation (3.15), this becomes

$$k_6\,\mathrm{d}\Phi = \frac{(\tilde{W}'\,\mathrm{d}\tilde{R} - \tilde{R}'\,\mathrm{d}\tilde{W})}{\sqrt{B^2 - 4AC}} \tag{4.7}$$

where

$$k_6 = \frac{k_5}{\sqrt{k_4}}.$$

Also, since

$$\frac{\partial \tilde{F}}{\partial \tilde{\rho}} \equiv (2A\tilde{\rho} + B) = \sqrt{B^2 - 4AC}$$

we can write

$$k_6\,\mathrm{d}\Phi = \frac{(\tilde{W}'\,\mathrm{d}\tilde{R} - \tilde{R}'\,\mathrm{d}\tilde{W})}{\partial\tilde{F}/\partial\tilde{\rho}}. \tag{4.8}$$

Generally speaking, if three Cartesian coordinates y_1, y_2, y_3 (to be identified with $\tilde{W}$, $\tilde{R}$ and $\tilde{\rho}$ respectively) obey a differential system and are submitted to a constraint

$$F(y_1, y_2, y_3) = 0$$

the general solution of this system must have the form

$$(\overrightarrow{\mathrm{grad}}\,F)\,\mathrm{d}\Phi = Q\,(\vec{y}' \wedge \mathrm{d}\vec{y}) \tag{4.9}$$

where Q may be called the integrating factor. Thus in the present case the integrating factor is a constant.

Finally, transforming back to coordinates

$$z_1 \equiv (W - W_0) \qquad z_2 \equiv (R - R_0) \qquad z_3 \equiv (\rho - \rho_0)$$

we find

$$(\tilde{W}'\,\mathrm{d}\tilde{R} - \tilde{R}'\,\mathrm{d}\tilde{W}) = \frac{1}{z_3^3}\,(\vec{z}, \vec{z}', \mathrm{d}\vec{z}) \tag{4.10}$$

where the triple product is also

$$(\vec{z}, \vec{z}', \mathrm{d}\vec{z}) = (\vec{z}\cdot\overrightarrow{\mathrm{grad}}\,F)\,\frac{(z_1'\,\mathrm{d}z_2 - z_2'\,\mathrm{d}z_1)}{\partial F/\partial z_3} \tag{4.11}$$

and $\overrightarrow{\mathrm{grad}} \equiv \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\right)$.

Further, as a consequence of the form of the transformation formulae (3.5)

$$\frac{\partial}{\partial\tilde{\rho}} \equiv -z_3(\vec{z}\cdot\overrightarrow{\mathrm{grad}}) \tag{4.12}$$

and the general solution is therefore

$$-k_6\,\mathrm{d}\Phi = \frac{1}{z_3^4}\,\frac{\partial F/\partial\tilde{\rho}}{\partial\tilde{F}/\partial\tilde{\rho}}\,\frac{\left(z_1'\,\mathrm{d}z_2 - z_2'\,\mathrm{d}z_1\right)}{\partial F/\partial z_3}. \tag{4.13}$$

Recalling $F \equiv z_3^4\,\tilde{F}$, and taking account of the constraint $F = \tilde{F} = 0$, we have

$$\frac{\partial F/\partial\tilde{\rho}}{\partial\tilde{F}/\partial\tilde{\rho}} = z_3^4$$

and hence

$$-k_6\,(\overrightarrow{\mathrm{grad}}\,F)\,\mathrm{d}\Phi = (\vec{x}' \wedge \mathrm{d}\vec{x}) \tag{4.14}$$

where $x_1 \equiv W, x_2 \equiv R, x_3 \equiv \rho$. Comparing with the general form (4.9) we see that *the integrating factor in the original coordinates is just a constant*.

These results confirm the central role of the matrix $\mathcal{M}$ (section 2.5), whose determinant is the function F, in both the formulation and the solution of the problem.

5. Conclusion

Through an appropriate generalization (sections 2.2–2.4) of the variables (ρ, W, R) introduced by Gaffet (1998) in cases without rotation, we have obtained a new formulation (section 2.5) in terms of a 4×4 symmetric matrix $\mathcal{M}$ which is a linear function of the four coordinates ρ, W, R and π (which is one of the two parameters defining the shape of the cloud). The matrix $\mathcal{M}$ is not regular, and the constraint

$$\det \mathcal{M} = 0$$

determines a surface (S) in phase space (Liouville torus) where all three integrals of motion remain constant. A geometrical study of the singular points (conic points) on that surface directly leads to a reformulation in terms of a pair of separable variables ℓ_1 and ℓ_2. The general form of the separable system (equation (3.19)) coincides with that found in non-rotating cases, except for the coefficients of the sixth-degree polynomial $m^2(l)$, which depend on the values of the three integrals of motion[2].

Using the particular (but generic) example $\varepsilon = 4$, $m = 6$; $\alpha^2 = 12$, the polynomial $m^2(\ell)$ has been found to coincide with a polynomial, denoted by A_6 (section 2.4), which can be obtained in a simple way through the analysis of the singular curve $\rho = 0$. As a result, the separable formulation (3.19) may be considered to be precisely established for all values of the three integrals of motion.

As pointed out in paper I (see section 7.4) the physical minimum of the energy constant, for given values of the other integrals ε and α^2, must correspond to a polynomial m^2 having a double root; these cases may thus be found by calculating the discriminant of m^2 and requiring it to vanish.

Appendix. The case $\varepsilon = 4$; $m = 6$; $\alpha^2 = 12$

The polynomial A_6, or $m^2(\ell)$, which is given by equation (3.16), has the following four real roots:

$$\begin{aligned} \ell_a &= 6.462\,40 \\ \ell_b &= -1.839\,69 \\ \ell_c &= -2.515\,06 \\ \ell_d &= -4.252\,77 \end{aligned} \tag{A.1}$$

and two complex conjugate roots, ℓ_e, ℓ_e^*.

Performing the linear transformation (3.5), which removes the conic point K_0 to infinity, the equation of the surface (S) assumes the quadratic form (3.7), where the coefficients A, B, C are the following polynomials in the new variables $\tilde{W}$ and $\tilde{R}$:

$$\begin{aligned} A(\tilde{W}, \tilde{R}) &= -57.7869 + (91.3625\,\hat{R} - 106.4472\,\tilde{W}) \\ &\quad + (-5.369\,97\,\hat{R}^2 + 60.1281\,\hat{R}\,\tilde{W} - 127.9787\,\tilde{W}^2) \\ B(\tilde{W}, \tilde{R}) &= 10.560\,30 + (18.338\,66\,\hat{R} + 53.8983\,\tilde{W}) \\ &\quad + (-2.129\,24\,\hat{R}^2 + 4.474\,29\,\hat{R}\,\tilde{W} + 54.4408\,\tilde{W}^2) \\ &\quad + (-0.280\,2187\,\hat{R}^3 + 5.598\,62\,\hat{R}^2\,\tilde{W} - 21.5690\,\hat{R}\,\tilde{W}^2 + 6.716\,04\,\tilde{W}^3) \\ C(\tilde{W}, \tilde{R}) &= \hat{R} - (0.8\,\hat{R}^2 + 8\,\hat{R}\,\tilde{W} + 14\,\tilde{W}^2) - (\hat{R}^3 + 1.6\,\hat{R}\,\tilde{W}^2 + 8\,\tilde{W}^3) \\ &\quad - \tilde{W}^2\,(\hat{R}^2 + 0.8\,\tilde{W}^2) \end{aligned} \tag{A.2}$$

[2] System (3.19) also coincides with the equations of motion for the Kowalevski top (Kowalevski 1889, 1890, Komarov 2001, Markushevich 2001), in which the polynomial m^2 is of fifth degree.

and we have introduced for convenience the rescaling

$$\hat{R} \equiv \frac{\tilde{R}}{4\,\varepsilon} = \frac{\tilde{R}}{16}.$$

The discriminant $B^2 - 4AC$ is found to be decomposable into six linear factors of the general form (3.8), with coefficients p_i, q_i which are related to the roots (A.1) of m^2 by equations (3.9), involving the following three quadratic polynomials P_2, Q_2, R_2:

$$\begin{aligned} P_2(\ell) &= 0.142\,958\,\ell^2 - 14.7252\,\ell - 41.1358 \\ Q_2(\ell) &= 0.768\,010\,\ell^2 - 1.832\,55\,\ell - 34.2347 \\ R_2(\ell) &= 1.374\,02\,\ell^2 + 3.826\,27\,\ell + 0.828\,790. \end{aligned} \tag{A.3}$$

References

Dyson F J 1968 *J. Math. Mech.* **18** 91
Gaffet B 1998 *J. Phys. A: Math. Gen.* **31** 8341
——2000 *J. Phys. A: Math. Gen.* **33** 3929
——2001 *J. Phys. A: Math. Gen.* **34** 2097
Komarov I V 2001 *J. Phys. A: Math. Gen.* **34** 2111
Kowalevski S 1889 *Acta Math.* **12** 177
——1890 *Acta Math.* **14** 81
Landau L D and Lifshitz E M 1960 *Mechanics, Course of Theoretical Physics* vol 1 (Oxford: Pergamon)
Markushevich D 2001 *J. Phys. A: Math. Gen.* **34** 2125
Ovsiannikov L V 1956 *Dokl. Akad. Nauk* **111** 47